9-1 Exponential Functions

Main Ideas

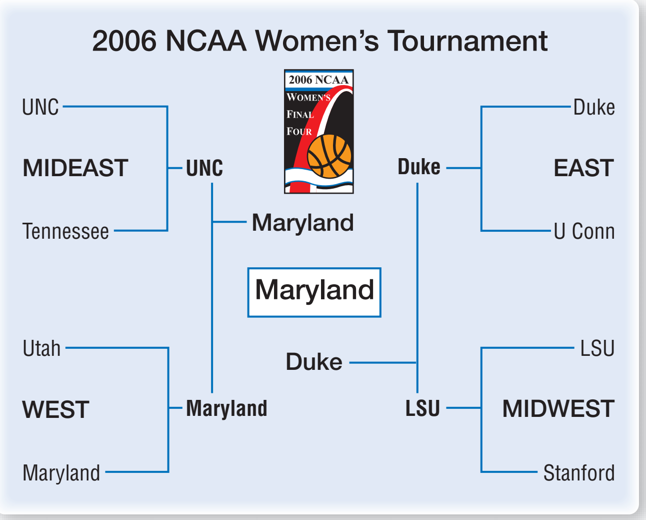
- Graph exponential functions.
- Solve exponential equations and inequalities.

New Vocabulary

exponential function
exponential growth
exponential decay
exponential equation
exponential inequality

GET READY for the Lesson

The NCAA women's basketball tournament begins with 64 teams and consists of 6 rounds of play. The winners of the first round play against each other in the second round. The winners then move from the Sweet Sixteen to the Elite Eight to the Final Four and finally to the Championship Game. The number of teams y that compete in a tournament of x rounds is $y = 2^x$.



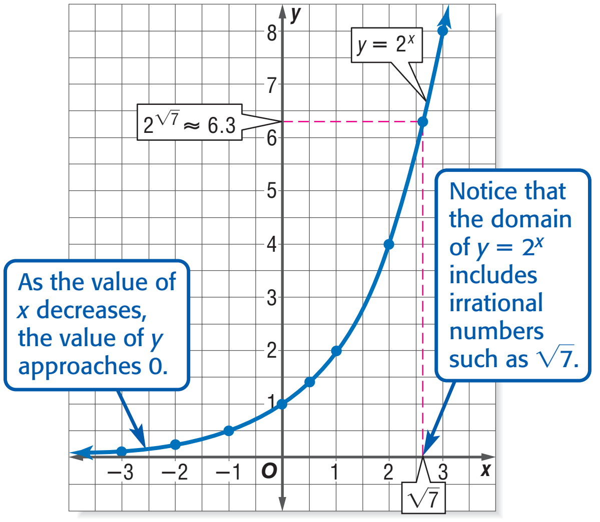
Exponential Functions In an exponential function like $y = 2^x$, the base is a constant, and the exponent is a variable. Let's examine the graph of $y = 2^x$.

EXAMPLE 1 Graph an Exponential Function

Sketch the graph of $y = 2^x$. Then state the function's domain and range.

Make a table of values. Connect the points to sketch a smooth curve.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2}$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



The domain is all real numbers, and the range is all positive numbers.

CHECK Your Progress

1. Sketch the graph of $y = \left(\frac{1}{2}\right)^x$. Then state the function's domain and range.

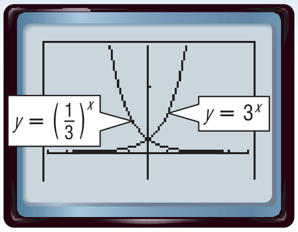
You can use a TI-83/84 Plus graphing calculator to look at the graphs of two other exponential functions, $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.

GRAPHING CALCULATOR LAB

Families of Exponential Functions

The calculator screen shows the graphs of parent functions $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.

THINK AND DISCUSS



[−5, 5] scl: 1 by [−2, 8] scl: 1

1. How do the shapes of the graphs compare?
2. How do the asymptotes and y-intercepts of the graphs compare?
3. Describe the relationship between the graphs.
4. Graph each group of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.
   a. $y = 2^x$, $y = 3^x$, and $y = 4^x$
   b. $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$
   c. $y = -3(2)^x$ and $y = 3(2)^x$; $y = -1(2)^x$ and $y = 2^x$.
5. Describe the relationship between the graphs of $y = -1(2)^x$ and $y = 2^x$. Then graph the functions on a graphing calculator to verify your conjecture.

Study Tip

Common Misconception

Be sure not to confuse polynomial functions and exponential functions. While $y = x^3$ and $y = 3^x$ each have an exponent, $y = x^3$ is a polynomial function and $y = 3^x$ is an exponential function.

The Graphing Calculator Lab allowed you to discover many characteristics of the graphs of exponential functions. In general, an equation of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, is called an **exponential function** with base b. Exponential functions have the following characteristics.

1. The function is continuous and one-to-one.
2. The domain is the set of all real numbers.
3. The x-axis is an asymptote of the graph.
4. The range is the set of all positive numbers if $a > 0$ and all negative numbers if $a < 0$.
5. The graph contains the point $(0, a)$. That is, the y-intercept is a.
6. The graphs of $y = ab^x$ and $y = a\left(\frac{1}{b}\right)^x$ are reflections across the y-axis.

Study Tip

Look Back

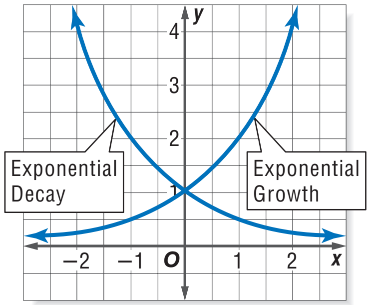
To review **continuous functions** and **one-to-one functions**, see Lessons 2-1 and 7-2.

Study Tip

Exponential Growth and Decay

Notice that the graph of an exponential growth function *rises* from left to right. The graph of an exponential decay function *falls* from left to right.

There are two types of exponential functions: **exponential growth** and **exponential decay**. The base of an exponential growth function is a number greater than one. The base of an exponential decay function is a number between 0 and 1.



KEY CONCEPT *Exponential Growth and Decay*

Symbols If $a > 0$ and $b > 1$, the function $y = ab^x$ represents exponential growth.

Example If $a > 0$ and $0 < b < 1$, the function $y = ab^x$ represents exponential decay.

EXAMPLE Identify Exponential Growth and Decay

2 **Determine whether each function represents exponential *growth* or *decay*.**

a. $y = \left(\frac{1}{5}\right)^x$

The function represents exponential decay, since the base, $\frac{1}{5}$, is between 0 and 1.

b. $y = 7(1.2)^x$

The function represents exponential growth, since the base, 1.2, is greater than 1.

CHECK Your Progress

2A. $y = 2(5)^x$

2B. $y = \left(\frac{2}{3}\right)^x$

Exponential functions are frequently used to model the growth or decay of a population. You can use the y-intercept and one other point on the graph to write the equation of an exponential function.

Study Tip

Checking Reasonableness

In Example 2, you learned that if $a > 1$ and $b > 1$, then the function represents growth. Here, $a = 1{,}321{,}045$ and $b = 1.002$, and the population representing growth increased.

Real-World EXAMPLE Write an Exponential Function

3 **POPULATION** **In 2000, the population of Phoenix was 1,321,045, and it increased to 1,331,391 in 2004.**

a. Write an exponential function of the form $y = ab^x$ that could be used to model the population y of Phoenix. Write the function in terms of x, the number of years since 2000.

For 2000, the time x equals 0, and the initial population y is 1,321,045. Thus, the y-intercept, and value of a, is 1,321,045.

For 2004, the time x equals 2004 – 2000 or 4, and the population y is 1,331,391. Substitute these values and the value of a into an exponential function to approximate the value of b.

$y = ab^x$	Exponential function
$1{,}331{,}391 = 1{,}321{,}045b^4$	Replace x with 4, y with 1,331,391, and a with 1,321,045.
$1.008 \approx b^4$	Divide each side by 1,321,045.
$\sqrt[4]{1.008} \approx b$	Take the 4th root of each side.

To find the 4^{th} root of 1.008, use selection 5: $\sqrt[x]{\ }$ under the MATH menu on the TI–83/84 Plus.

KEYSTROKES: 4 [MATH] 5 1.008 [ENTER] 1.001994028

An equation that models the population growth of Phoenix from 2000 to 2004 is $y = 1{,}321{,}045(1.002)^x$.

b. Suppose the population of Pheonix continues to increase at the same rate. Estimate the population in 2015.

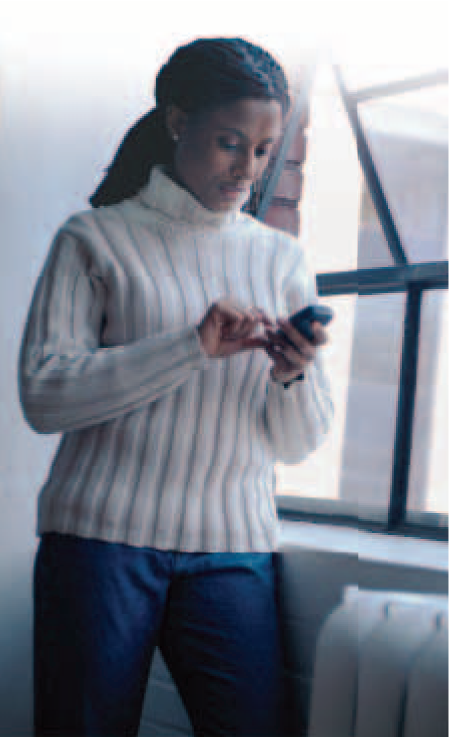
For 2015, the time equals 2015 – 2000 or 15.

$y = 1{,}321{,}045(1.002)^x$ Modeling equation

$= 1{,}321{,}045(1.002)^{15}$ Replace x with 15.

$\approx 1{,}360{,}262$ Use a calculator.

The population in Phoenix will be about 1,360,262 in 2015.



Real-World Link

The first virus that spread via cell phone networks was discovered in June 2004.

Source: internetnews.com

CHECK Your Progress

3. SPAM In 2003, the amount of annual cell phone spam messages totaled about ten million. In 2005, the total grew exponentially to 500 million. Write an exponential function of the form $y = ab^x$ that could be used to model the increase of spam messages y. Write the function in terms of x, the number of years since 2003. If the number of spam messages continues increasing at the same rate, estimate the annual number of spam messages in 2010.

Online Personal Tutor at algebra2.com

Exponential Equations and Inequalities

Exponential equations are equations in which variables occur as exponents.

KEY CONCEPT *Property of Equality for Exponential Functions*

Symbols If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example If $2^x = 2^8$, then $x = 8$.

EXAMPLE 4 Solve Exponential Equations

Solve each equation.

a. $3^{2n+1} = 81$

$3^{2n+1} = 81$ Original equation

$3^{2n+1} = 3^4$ Rewrite 81 as 3^4 so each side has the same base.

$2n + 1 = 4$ Property of Equality for Exponential Functions

$2n = 3$ Subtract 1 from each side.

$n = \frac{3}{2}$ Divide each side by 2.

(continued on the next page)

CHECK $3^{2n+1} = 81$ Original equation

$3^{2\left(\frac{3}{2}\right)+1} \stackrel{?}{=} 81$ Substitute $\frac{3}{2}$ for n.

$3^4 \stackrel{?}{=} 81$ Simplify.

$81 = 81$ ✔ Simplify.

b. $4^{2x} = 8^{x-1}$

$4^{2x} = 8^{x-1}$ Original equation

$(2^2)^{2x} = (2^3)^{x-1}$ Rewrite each side with a base of 2.

$2^{4x} = 2^{3(x-1)}$ Power of a Power

$4x = 3(x-1)$ Property of Equality for Exponential Functions

$4x = 3x - 3$ Distributive Property

$x = -3$ Subtract $3x$ from each side.

CHECK $4^{2x} = 8^{x-1}$ Original equation

$4^{2(-3)} \stackrel{?}{=} 8^{-3-1}$ Substitute -3 for x.

$4^{-6} \stackrel{?}{=} 8^{-4}$ Simplify.

$\frac{1}{4096} = \frac{1}{4096}$ ✔ Simplify.

CHECK Your Progress

Solve each equation.

4A. $4^{2n-1} = 64$

4B. $5^{5x} = 125^{x+2}$

The following property is useful for solving inequalities involving exponential functions or **exponential inequalities**.

KEY CONCEPT ***Property of Inequality for Exponential Functions***

Symbols If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Example If $5^x < 5^4$, then $x < 4$.

This property also holds true for $\leq$ and $\geq$.

EXAMPLE 5 Solve Exponential Inequalities

Study Tip

Look Back

You can review **negative exponents** in Lesson 6-1.

5 Solve $4^{3p-1} > \frac{1}{256}$.

$4^{3p-1} > \frac{1}{256}$ Original inequality

$4^{3p-1} > 4^{-4}$ Rewrite $\frac{1}{256}$ as $\frac{1}{4^4}$ or 4^{-4} so each side has the same base.

$3p - 1 > -4$ Property of Inequality for Exponential Functions

$3p > -3$ Add 1 to each side.

$p > -1$ Divide each side by 3.

CHECK Test a value of p greater than -1; for example, $p = 0$.

$$4^{3p - 1} > \frac{1}{256} \quad \text{Original inequality}$$

$$4^{3(0) - 1} \overset{?}{>} \frac{1}{256} \quad \text{Replace } p \text{ with } 0.$$

$$4^{-1} \overset{?}{>} \frac{1}{256} \quad \text{Simplify.}$$

$$\frac{1}{4} > \frac{1}{256} \checkmark \quad a^{-1} = \frac{1}{a}$$

CHECK Your Progress

Solve each inequality.

5A. $3^{2x - 1} \geq \frac{1}{243}$ **5B.** $2^{x + 2} > \frac{1}{32}$

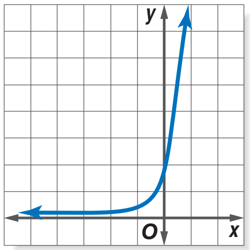
CHECK Your Understanding

Example 1 (p. 498–499)

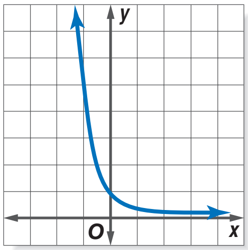
Match each function with its graph.

1. $y = 5^x$ **2.** $y = 2(5)^x$ **3.** $y = \left(\frac{1}{5}\right)^x$

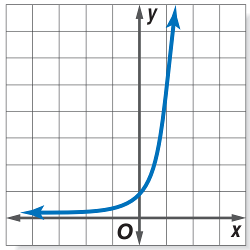
a.



b.



c.



Sketch the graph of each function. Then state the function's domain and range.

4. $y = 3(4)^x$ **5.** $y = 2\left(\frac{1}{3}\right)^x$

Example 2 (p. 500)

Determine whether each function represents exponential *growth* or *decay*.

6. $y = (0.5)^x$ **7.** $y = 0.3(5)^x$

Example 3 (pp. 500–501)

Write an exponential function for the graph that passes through the given points.

8. $(0, 3)$ and $(-1, 6)$ **9.** $(0, -18)$ and $(-2, -2)$

MONEY For Exercises 10 and 11, use the following information.
In 1993, My-Lien inherited \$1,000,000 from her grandmother. She invested all of the money, and by 2005, the amount had grown to \$1,678,000.

10. Write an exponential function that could be used to model the money y. Write the function in terms of x, the number of years since 1993.

11. Assume that the amount of money continues to grow at the same rate. Estimate the amount of money in 2015. Is this estimate reasonable? Explain your reasoning.

Example 4 (pp. 501–502)

Solve each equation. Check your solution.

12. $2^{n+4} = \frac{1}{32}$ **13.** $9^{2y-1} = 27^y$ **14.** $4^{3x+2} = \frac{1}{256}$

Example 5 (pp. 502–503)

Solve each inequality. Check your solution.

15. $5^{2x+3} \leq 125$ **16.** $3^{3x-2} > 81$ **17.** $4^{4a+6} \leq 16^a$

Exercises

HOMEWORK HELP

For Exercises	See Examples
18–21	1
22–27	2
28–38	3
39–44	4
45–48	5

Sketch the graph of each function. Then state the function's domain and range.

18. $y = 2(3)^x$ **19.** $y = 5(2)^x$

20. $y = 0.5(4)^x$ **21.** $y = 4\left(\frac{1}{3}\right)^x$

Determine whether each function represents exponential *growth* or *decay*.

22. $y = 10(3.5)^x$ **23.** $y = 2(4)^x$ **24.** $y = 0.4\left(\frac{1}{3}\right)^x$

25. $y = 3\left(\frac{5}{2}\right)^x$ **26.** $y = 30^{-x}$ **27.** $y = 0.2(5)^{-x}$

Write an exponential function for the graph that passes through the given points.

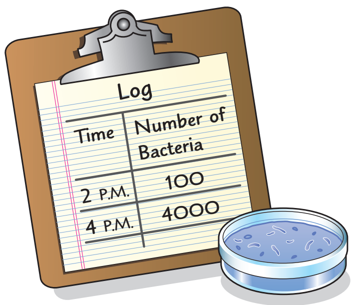
28. (0, −2) and (−2, −32) **29.** (0, 3) and (1, 15)

30. (0, 7) and (2, 63) **31.** (0, −5) and (−3, −135)

32. (0, 0.2) and (4, 51.2) **33.** (0, −0.3) and (5, −9.6)

BIOLOGY For Exercises 34 and 35, use the following information.

The number of bacteria in a colony is growing exponentially.



34. Write an exponential function to model the population y of bacteria x hours after 2 P.M.

35. How many bacteria were there at 7 P.M. that day?

Cross-Curricular Project

Math Online The magnitude of an earthquake can be represented by an exponential equation. Visit algebra2.com to continue work on your project.

MONEY For Exercises 36–38, use the following information.

Suppose you deposit a principal amount of P dollars in a bank account that pays compound interest. If the annual interest rate is r (expressed as a decimal) and the bank makes interest payments n times every year, the amount of money A you would have after t years is given by $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

36. If the principal, interest rate, and number of interest payments are known, what type of function is $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$? Explain your reasoning.

37. Write an equation giving the amount of money you would have after t years if you deposit \$1000 into an account paying 4% annual interest compounded quarterly (four times per year).

38. Find the account balance after 20 years.

EXTRA PRACTICE
See pages 909, 934.

Math Online
Self-Check Quiz at algebra2.com

Solve each equation. Check your solution.

39. $2^{3x+5} = 128$ **40.** $5^{n-3} = \frac{1}{25}$ **41.** $\left(\frac{1}{9}\right)^m = 81^{m+4}$

42. $\left(\frac{1}{7}\right)^{y-3} = 343$ **43.** $10^{x-1} = 100^{2x-3}$ **44.** $36^{2p} = 216^{p-1}$

Solve each inequality. Check your solution.

45. $3^{n-2} > 27$ **46.** $2^{2n} \leq \frac{1}{16}$

47. $16^n < 8^{n+1}$ **48.** $32^{5p+2} \geq 16^{5p}$

Sketch the graph of each function. Then state the function's domain and range.

49. $y = -\left(\frac{1}{5}\right)^x$ **50.** $y = -2.5(5)^x$

Real-World Link
Since computers were invented, computational speed has multiplied by a factor of 4 about every three years.

Source: wired.com

COMPUTERS **For Exercises 51 and 52, use the information at the left.**

51. If a typical computer operates with a computational speed s today, write an expression for the speed at which you can expect an equivalent computer to operate after x three-year periods.

52. Suppose your computer operates with a processor speed of 2.8 gigahertz and you want a computer that can operate at 5.6 gigahertz. If a computer with that speed is currently unavailable for home use, how long can you expect to wait until you can buy such a computer?

POPULATION **For Exercises 53–55, use the following information.**
Every ten years, the Bureau of the Census counts the number of people living in the United States. In 1790, the population of the U.S. was 3.93 million. By 1800, this number had grown to 5.31 million.

53. Write an exponential function that could be used to model the U.S. population y in millions for 1790 to 1800. Write the equation in terms of x, the number of decades x since 1790.

54. Assume that the U.S. population continued to grow at least that fast. Estimate the population for the years 1820, 1840, and 1860. Then compare your estimates with the actual population for those years, which were 9.64, 17.06, and 31.44 million, respectively.

55. **RESEARCH** Estimate the population of the U.S. in the most recent census. Then use the Internet or other reference to find the actual population of the U.S. in the most recent census. Has the population of the U.S. continued to grow at the same rate at which it was growing in the early 1800s? Explain.

Graphing Calculator

Graph each pair of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.

	Parent Function	New Function		Parent Function	New Function
56.	$y = 2^x$	$y = 2^x + 3$	**57.**	$y = 3^x$	$y = 3^{x+1}$
58.	$y = \left(\frac{1}{5}\right)^x$	$y = \left(\frac{1}{5}\right)^{x-2}$	**59.**	$y = \left(\frac{1}{4}\right)^x$	$y = \left(\frac{1}{4}\right)^x - 1$

60. Describe the effect of changing the values of h and k in the equation $y = 2^{x-h} + k$.

H.O.T. Problems

61. **OPEN ENDED** Give an example of a value of b for which $y = b^x$ represents exponential decay.

Jeff Zaruba/CORBIS

62. REASONING Identify each function as *linear*, *quadratic*, or *exponential*.

a. $y = 3x^2$ **b.** $y = 4(3)^x$ **c.** $y = 2x + 4$ **d.** $y = 4(0.2)^x + 1$

63. CHALLENGE Decide whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For a positive base b other than 1, $b^x > b^y$ if and only if $x > y$.

64. *Writing in Math* Use the information about women's basketball on page 498 to explain how an exponential function can be used to describe the teams in a tournament. Include an explanation of how you could use the equation $y = 2^x$ to determine the number of rounds of tournament play for 128 teams and an example of an inappropriate number of teams for a tournament.

STANDARDIZED TEST PRACTICE

65. ACT/SAT If $4^{x+2} = 48$, then $4^x =$

A 3.0

B 6.4

C 6.9

D 12.0

66. REVIEW If the equation $y = 3^x$ is graphed, which of the following values of x would produce a point closest to the x-axis?

F $\frac{3}{4}$

G $\frac{1}{4}$

H 0

J $-\frac{3}{4}$

Spiral Review

Solve each equation. Check your solutions. (Lesson 8-6)

67. $\frac{15}{p} + p = 16$ **68.** $\frac{s-3}{s+4} = \frac{6}{s^2-16}$ **69.** $\frac{2a-5}{a-9} + \frac{a}{a+9} = \frac{-6}{a^2-81}$

Identify each equation as a type of function. Then graph the equation. (Lesson 8-5)

70. $y = \sqrt{x-2}$ **71.** $y = -2[\![x]\!]$ **72.** $y = 8$

Find the inverse of each matrix, if it exists. (Lesson 4-7)

73. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **74.** $\begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$ **75.** $\begin{bmatrix} -5 & 6 \\ -11 & 3 \end{bmatrix}$

76. ENERGY A circular cell must deliver 18 watts of energy. If each square centimeter of the cell that is in sunlight produces 0.01 watt of energy, how long must the radius of the cell be? (Lesson 7-4)

GET READY for the Next Lesson

Find $g[h(x)]$ and $h[g(x)]$. (Lesson 7-5)

77. $h(x) = 2x - 1$, $g(x) = x - 5$

78. $h(x) = x + 3$, $g(x) = x^2$

79. $h(x) = 2x + 5$, $g(x) = -x + 3$